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UNIVALENCE CRITERIA AND THE HYPERBOLIC METRIC

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1. Introduction. We shall consider restrictions on the derivative of a function $f \in H(\mathbf{B})$ (i.e., holomorphic in the unit disk $\mathbf{B}$) which imply that f is univalent. Perhaps the best known result of this type, due to Wolff [13], Warschawski [12] and Noshiro [11], involves only the argument of f'. It states that f is one-to-one if $f'(z) \neq 0$ and $\arg f'(z)$ lies in an interval of length π, $z \in \mathbf{B}$. If the length of the interval is larger than π, then f need not be univalent, and, in fact, the valence of f need not be bounded [6].

On the other hand, there is a criterion for univalence due to John [7] which involves only the modulus of f'. For non-constant $f \in H(\mathbf{B})$ let $M_f = \sup_{z\in\mathbf{B}}|f'(z)|$, $m_f = \inf_{z\in\mathbf{B}}|f'(z)|$ and $\mu_f = M_f/m_f$. The John constant γ is defined by $\gamma = \sup\{t: \mu_f \leqq t \text{ implies } f \text{ is univalent}\}$. If $\mu_f \leqq \gamma$, then f is univalent.

The condition $\mu_f < \infty$ is equivalent to $f'(\mathbf{B})$ lying in an annulus centered at zero. We may introduce symmetry relative to the unit circle by considering $g = f/\sqrt{m_f M_f}$. Then $M_g = \sqrt{M_f/m_f} = 1/m_g$, $\mu_g = \mu_f$, and, of course, f is univalent if and only if g is. It follows that

$$\frac{1}{2}\log\gamma = \sup\{M: e^{-M} \leqq |f'| \leqq e^M \Rightarrow f \text{ is univalent}\}.$$

The best known estimates for γ are $e^{\pi/2} \leqq \gamma \leqq e^{\pi}$; the lower and upper bounds being given by John [8] and Yamashita [14], respectively.

In the next section we consider the problem of determining which plane regions Ω have the property that $\log f'(\mathbf{B}) \subset \Omega$ implies f is univalent. The above two criteria correspond to the cases in which Ω is a horizontal or vertical strip, respectively. We obtain conditions on Ω, involving the hyperbolic metric on Ω, which insure that f is one-one. Our results rely on the following theorem due to Becker [3].

BECKER'S UNIVALENCE CRITERION. *If $f \in H(\mathbf{B})$, $f'(0) \neq 0$, and*

$$(1 - |z|^2)\left|\frac{zf''(z)}{f'(z)}\right| \leqq 1, \; z \in \mathbf{B},$$

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